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PASCAL LINE EQUATIONS AND SOME CONSEQUENCES.

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I know of no attempt to give the equations of the Pascal lines of the inscribed hexagon of a conic except that made by Salmon in Articles 267–8 of his *Conic Sections*, and this does not reach any definite form.

Salmon takes *abcdef* as the hexagon order; $ab = 0$, as the equation of *ab* (*specifying no particular form of equation*).

The conic circumscribing *abcd*, *defa* is either

$$ab\cdot cd = ad\cdot bc \tag{1}$$

or,

$$de\cdot fa = ad\cdot ef; \tag{2}$$

$$\therefore\quad ab\cdot cd - de\cdot fa = ad(bc - ef). \tag{3}$$

The left hand side of (3) (a conic about $ab - de - cd - af$) thus splits into two factors; one is the diagonal *ad*; therefore the other, $bc - ef = 0$, is the other diagonal, or the Pascal line, since (3) shows that the points *ab*, *de*; *cd*, *fa*; *bc*, *ef*, are on this line.

All that this shows is that the equation of the Pascal may be expressed in terms of those of a pair of opposite sides, and that for any particular hexagon and conic it could be gotten by an appropriate process of subtraction and factoring.

Clearly, Salmon's result (the difference of the equations of a pair of opposite sides) would not, in general, hold if the equations were in "normal form."

Moreover if Salmon's process in Art. 268, to prove concurrency of Pascal lines in a Steiner's g-point, is applied to a Kirkman's h-point, *we may have exactly the same sets of equations to prove concurrency for two entirely different sets of lines* (one line in common).

Consider the g-point and h-point:[1]

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[1] This notation is different from Salmon's and better. Salmon indicates the Pascal lines by pairs of horizontal lines, $\left(\frac{ab}{ed}\right)$ indicating the point of intersection of lines *ab*, *ed*, but his horizontal lines do not give the hexagons involved. In both (g) and (h) below, the horizontal lines give the hexagons involved, and, taken in pairs, the Pascal lines of these hexagons. In (g), line (2) is formed from line (1) by writing under each segment of (1) its opposite segment in (1), reversing, in each segment, the hexagon order of letters in (1); (3) from (2) as (2) from (1); lines (1), (2) indicate the Pascal of (1); (2), (3) that of (2); (3), (1) that of (3). In (h) the same procedure is followed, except that the hexagon order is retained in one column (in this particular h-point, the third) and reversed in the other two; (1), (2) give the Pascal of (1); (2), (3) that of (2); (3), (1′) that of (3); (1′) is (1).

In this notation also the quadrilaterals to which to refer the conic to prove concurrency for (g), (h) are indicated. For concurrency of (g) the Pascals may be expressed in terms of the opposite sides in any column, and the reference quadrilaterals are the lines of the two remaining columns. For concurrency of (h), we must express the Pascals in terms of the sides of the 3d column (always that column in which the hexagon order is held) and the reference quadrilaterals are the lines of the first and second columns. Note now that the third columns of (g) and (h) indi-

$$\begin{bmatrix} ab, & cd, & ef \\ ed, & af, & cb \\ cf, & eb, & ad \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \quad (g); \qquad \begin{bmatrix} ab, & cd, & ef \\ ed, & af, & bc \\ bf, & ec, & da \\ dc, & ba, & fe \end{bmatrix} \begin{matrix} (1) \\ (5) \\ (6) \\ (1') \end{matrix} \quad (h).$$

Selecting as reference quadrilaterals for (g)

$$abcd, \quad edaf, \quad cfeb,$$

the conic is

$$\left.\begin{array}{l} ab\cdot cd = ad\cdot bc \quad (1), \\ \text{or,} \\ ed\cdot af = ad\cdot ef \quad (2), \\ \text{or,} \\ cf\cdot eb = ef\cdot bc \quad (3). \end{array}\right\} \qquad (A)$$

Subtracting in pairs

$$\left.\begin{array}{l} ab\cdot cd - ed\cdot af = ad\,(bc - ef), \\ ed\cdot af - cf\cdot eb = ef\,(ad - bc), \\ cf\cdot eb - ab\cdot cd = bc\,(ef - ad). \end{array}\right\} \qquad (B)$$

And the Pascals are,

$$bc - ef = 0, \qquad (4)$$

$$ad - bc = 0, \qquad (5)$$

$$ef - ad = 0. \qquad (6)$$

Identifying the conic with quadrilaterals in the first and second columns of (h), the conic is

$$ab\cdot cd = ad\cdot bc \qquad (7)$$

or

$$ed\cdot af = ad\cdot ef \qquad (8)$$

or

$$bf\cdot ec = ef\cdot bc. \qquad (9)$$

On subtracting in pairs,

$$\left.\begin{array}{l} ab\cdot cd - ed\cdot af = ad\,(bc - ef), \\ ed\cdot af - bf\cdot ec = ef\,(ad - bc), \\ bf\cdot ec - ab\cdot cd = bc\,(ef - ad). \end{array}\right\} \qquad (D)$$

And again the Pascals are,

$$bc - ef = 0, \qquad (10)$$

$$ad - bc = 0, \qquad (11)$$

$$ef - ad = 0, \qquad (12)$$

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cate *the same set of opposite sides.* Thus, Steiner's theorem, as expressed in a g-point, and Kirkman's theorem, as expressed in an h-point, are capable of a single expression, namely, *if three hexagons have for pairs of opposite sides the sides of a triangle, taken in pairs, the Pascal lines of these hexagons are concurrent.* Concerning duality of properties of the complete Pascal hexagon see account of investigation by G. Veronese on pages x–xii of Vol. II, 6th ed. of Salmon-Fiedler's *Kegelschnitte.* Concerning advantages of this notation, see "Hexagon Notation" by R. D. Bohannan, in *Ohio Journal of Science* for February, 1916.

concurrent lines, with the same equations as (4), (5), (6), while (11), (12) are entirely different lines from (5), (6). (B) shows that (5) is the line *ed, cf*; *af, eb*; *bc, ad*; while (D) shows that (11), with the same equation as (5), is the line *ed, bf*; *af, ec*; *bc, ad*. And (B) shows that (6) is the line *cf, ab*; *eb, cd*; *ef, ad*, while (D) shows that (12), with the same equation as (6), is the line *bf, dc*; *ec, ab*; *ef, ad*.

The trouble is that if $ab = 0$ is the equation of ab, so is $k \cdot ab = 0$.

If equations in normal form (or any other specified form) are used, different multipliers are necessary to render the equations (A) identical; likewise for those in (C). If l_1, l_2, l_3 are such multipliers for (A), the Pascals are

$$l_1 \cdot bc - l_2 \cdot ef = 0,$$

$$l_2 \cdot ad - l_3 \cdot bc = 0,$$

$$l_3 \cdot ef - l_1 \cdot ad = 0,$$

concurrent lines.

Now a different set of multipliers would be used in (C) and thus different equations would result therefrom.

If $ab = 0$ is the equation in *normal form* of ab (and all other sides of the hexagon are assumed to be expressed in *normal form*), the equation of the Pascal line of

$$fabcde \tag{1}$$

may be written

$$f\begin{pmatrix} ad \\ ab \end{pmatrix} \cdot ab = c\begin{pmatrix} ad \\ de \end{pmatrix} \cdot de, \tag{p}$$

where $f\begin{pmatrix} ad \\ ab \end{pmatrix}$ denotes the ratio of the perpendiculars from f on ad and ab.

For (p) is evidently a line passing through the intersection of *ab, de*, a pair of opposite sides of (1). And (p) may also be written:

$$f\begin{pmatrix} ad \\ ab \end{pmatrix} \cdot ab - ad = c\begin{pmatrix} ad \\ de \end{pmatrix} \cdot de - ad,$$

which represents a line passing through the intersection of *af, cd*, another pair of opposite sides of (1).

(The manner of writing (p) from (1) is evident at a glance.)

If (1) is reversed and written

$$c\,b\,a\,f\,e\,d$$

its Pascal (in terms of *ba, ed*) is by (p),

$$c\begin{pmatrix} be \\ ba \end{pmatrix} \cdot ba = f\begin{pmatrix} be \\ ed \end{pmatrix} \cdot ed \tag{p'}$$

But (p') is identically the same as (p). For if the conic is

$$ab \cdot ed = m \cdot be \cdot da,$$

then

$$f\binom{ab}{ad} = mf\binom{be}{ab}$$

and

$$c\binom{de}{ad} = mc\binom{be}{ba}.$$

Thus the ratios of the coefficients in (p), (p') are the same.

Thus the equation of the Pascal of any hexagon may be expressed definitely in terms of any pair of opposite sides by (p). (Three equations of the same form.)

Taking the hexagons of the g-point

$$\begin{bmatrix} ab, & cd, & ef \\ ed, & af, & cb \\ cf, & eb, & ad \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \qquad (g)$$

in the forms

$$bcdefa, \qquad (1)$$

$$dafcbe, \qquad (2)$$

$$febadc, \qquad (3)$$

and applying (p) to each form, we have, as the equations of the Pascals of (1), (2), (3),

$$b\binom{cf}{cd}\cdot cd = e\binom{cf}{fa}\cdot fa, \qquad \text{(I)}$$

$$d\binom{ab}{af}\cdot af = c\binom{ab}{be}\cdot be, \qquad \text{(II)}$$

$$f\binom{ed}{eb}\cdot eb = a\binom{ed}{dc}\cdot dc. \qquad \text{(III)}$$

But (I), (II), (III) are concurrent.

$$\therefore \quad \frac{a(ed)\cdot b(dc)\cdot e(cf)\cdot d(fa)\cdot c(ab)\cdot f(be)}{a(cd)\cdot f(de)\cdot c(eb)\cdot d(ba)\cdot e(af)\cdot b(fc)} = \text{unity}, \qquad (E)$$

where $a(ed)$ denotes the perpendicular from a on ed.

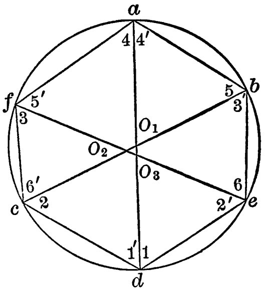
The hexagon, *abedcf*, appearing in the numerator and denominator here, is one of the three hexagons of which the selected point is the g-point. If we had expressed the Pascals in terms of the equations of pairs of opposite sides shown in any column of (g) other than the second (as above), another hexagon of which (g) is the g-point would have appeared. In the notation used in (g) the hexagons of which (g) is the g-point are shown in the columns, namely, *ab, ed, cf*; *cd, af, eb*; *ef, cb, ad*.

By (E) the g-point gives this theorem for any hexagon inscribed in a conic:

The product of the perpendiculars from each vertex on the second side ahead (that through the opposite vertex) in going around the hexagon in one direction, is equal to the product of the perpendiculars on the second side ahead going in the opposite direction.

And there are the usual variations for the pentagon and quadrilateral by letting two vertices of the hexagon be replaced by a tangent.

In the case of a triangle the perpendiculars going one way are the same as those going the other way.



This theorem is self-evident for a circle since it is equivalent to

$$\frac{\sin\angle 1\cdot\sin\angle 2\cdot\sin\angle 3\cdot\sin\angle 4\cdot\sin\angle 5\cdot\sin\angle 6}{\sin\angle 5'\cdot\sin\angle 4'\cdot\sin\angle 3'\cdot\sin\angle 2'\cdot\sin\angle 1'\cdot\sin\angle 6'} = \text{unity} \qquad (E') \quad \text{(see the Fig.)},$$

in which equal angles are the one under the other. Thus (E), (E') are projective relations, (E') remaining true in any conic, where the angles are not equal in pairs. The reason is that this theorem can be proved directly as a consequence of the well-known theorem, "*if from each of two points pairs of parallel secants of a conic are drawn, the ratio of the rectangle of secants from the first point is the same as that from the second*" (F). And this is a projective relation.

Draw the hexagon, *abedcf*, in this order, on a conic. Let the lines joining opposite vertices meet: *ad*, *bc* in O_1; *bc*, *ef* in O_2; *ef*, *ad* in O_3.

$$a(de) = ad\cdot\sin\angle 1,$$
$$d(fa) = ad\cdot\sin\angle 4,$$
$$b(dc) = bc\cdot\sin\angle 2,$$
$$c(ab) = bc\cdot\sin\angle 5,$$
$$e(cf) = ef\cdot\sin\angle 3,$$
$$f(be) = ef\cdot\sin\angle 6.$$

Treat the denominator in the same way and (E) will finally reduce to

$$\frac{O_1a\cdot O_1d}{O_3a\cdot O_3d}\cdot\frac{O_3e\cdot O_3f}{O_2e\cdot O_2f}\cdot\frac{O_2b\cdot O_2c}{O_1b\cdot O_1c}. \tag{G}$$

Draw a line through O_1 parallel to $O_2O_3(ef)$, cutting the conic in A, B.

$$\therefore\ \frac{O_1a\cdot O_1d}{O_1A\cdot O_1B}=\frac{O_3a\cdot O_3d}{O_3e\cdot O_3f}$$

and

$$\frac{O_1A\cdot O_1B}{O_1b\cdot O_1c}=\frac{O_2e\cdot O_2f}{O_2c\cdot O_2b}.$$

Eliminate $O_1A\cdot O_1B$. The result is

$$(E) = \text{unity}.$$

(G) may be written thus:

$$\frac{O_1a\cdot O_1d\cdot O_2b\cdot O_2c\cdot O_3e\cdot O_3b}{O_1b\cdot O_1c\cdot O_2e\cdot O_2f\cdot O_3a\cdot O_3d}=1 \tag{G'}$$

and formulated into a proposition: "*if the opposite vertices of a hexagon are joined and the lines meet in three points the ratio of the product of rectangles as given in* (G') *is unity;* and corresponding propositions for the pentagon and quadrilateral.

Thus Steiner's theorem in the (g) point is a consequence of the theorem (F).

If we take the h-point

$$\begin{array}{l} \left[\begin{array}{lll} ab, & cd, & ef \\ ed, & fa, & cb \\ ca, & be, & fd \\ fe, & dc, & ba \end{array}\right] \end{array} \begin{array}{l} (1) \\ (2) \\ (3) \\ (1') \end{array} \tag{H}$$

and write the hexagons in the order

$$bcdefa \tag{1}$$

$$dfacbe \tag{2}$$

$$abefdc \tag{3}$$

and apply (p) to these, the Pascals are

$$b\binom{cf}{cd}\cdot cd = e\binom{cf}{fa}\cdot fa,$$

$$d\binom{fb}{fa}\cdot fa = c\binom{fb}{be}\cdot be,$$

$$a\binom{bd}{be}\cdot be = f\binom{bd}{dc}\cdot dc,$$

and these lines are concurrent.

$$\therefore\ \left\{\frac{a(be)\cdot d(fa)\cdot b(cd)}{e(fa)\cdot c(be)\cdot f(dc)}\right\}\cdot\left\{\frac{f(bd)\cdot c(fb)\cdot e(cf)}{b(cb)\cdot d(fb)\cdot a(bd)}\right\}=\text{unity}. \tag{I}$$

The hexagon appearing here (outside the parentheses) is *adbfce*, the hexagon to which (H) is unique (none of its sides appear in (H)); *be*, *fa*, *cd* appearing in parentheses in the first fraction are diagonals joining opposite vertices. And the proposition in (I) is this:

If we go half way around a hexagon dropping perpendiculars from the vertices on the second diagonal ahead and the rest of the way dropping perpendiculars from the vertices on the side next behind; and then reverse, going in the opposite direction, the products of the perpendiculars in the two circuits are equal.

If we write the hexagons of (H)

$$edcbaf \tag{1}$$

$$cafdeb \tag{2}$$

$$febacd \tag{3}$$

and apply (p), the Pascals are

$$e\begin{pmatrix} da \\ dc \end{pmatrix} \cdot dc = b\begin{pmatrix} da \\ af \end{pmatrix} \cdot af,$$

$$c\begin{pmatrix} ae \\ af \end{pmatrix} \cdot af = d\begin{pmatrix} ae \\ eb \end{pmatrix} \cdot eb,$$

$$f\begin{pmatrix} ec \\ eb \end{pmatrix} \cdot eb = a\begin{pmatrix} ec \\ cd \end{pmatrix} \cdot cd,$$

$$\therefore \quad \left\{ \frac{a(ce)\cdot d(ea)\cdot b(ad)}{e(da)\cdot c(ae)\cdot f(ec)} \right\} \left\{ \frac{f(eb)\cdot c(af)\cdot e(dc)}{b(af)\cdot d(eb)\cdot a(cd)} \right\} = \text{unity}. \tag{J}$$

In (J) we have, outside the parentheses, the same hexagon as in (I), *adbfce*, and the proposition of (I) is reversed:

If we go half way around a hexagon dropping perpendiculars from each vertex on the side immediately behind, and the remaining way on the second diagonal ahead; and then reverse, repeating, in the opposite direction, the products are equal.

We have previously shown that (I) and (J) are identical.

If we multiply (I) by (J), the product of the first fraction of (J) by the second of (I) is evidently unity.

$$\therefore \quad \frac{a(be)\cdot e(cd)\cdot c(fa)\cdot f(be)\cdot b(dc)\cdot d(ab)}{a(dc)\cdot d(be)\cdot b(fa)\cdot f(cd)\cdot c(eb)\cdot e(af)} = \text{unity}. \tag{K}$$

And (K) is this:

If we go around a hexagon (here aecfbd) dropping perpendiculars on the diagonal (of opposite vertices) just ahead, and then reverse, the product of the perpendiculars in the one circuit is that of the other.

(We might have said "second diagonal ahead," and then this proposition is the same as that for a g-point, replacing the word "side" by "diagonal".)

This theorem can also be readily proven directly. If the hexagon, *adbfce*, is drawn on a conic, and the perpendiculars expressed in terms of diagonals and sines of angles, identically the same angle relation as expressed in (E') will appear.

Thus not only are Steiner's theorem in a (g) point and Kirkman's theorem in an (h) point capable of expression in the same form, but they represent identically the same relation[1] among sines of angles between the sides and diagonals of the conic, namely (E'),—a relation of equality of sines in the case of the circle (as in (E')).

Another form of Pascal equation is, for *fabcde* (1),

$$\frac{ab}{ad}\cdot f\binom{ad}{ab}+\frac{cd}{ad}\cdot e\binom{ad}{cd}=1. \tag{q_1}$$

For (q_1) is evidently a line through the intersection of $ab = 0$ and $ad - cd\cdot e\binom{ad}{cd} = 0$, or, of ab and de, a pair of opposite sides of (1). It also passes through the intersection of $cd = 0$, and $ad - ab\cdot f\binom{ad}{ab} = 0$, or, of cd and af, another pair of opposite sides of (1).

The third line of (g) may be written

$$ebadcf \tag{3}$$

and its Pascal is by (q_1),

$$\frac{ba}{bc}\cdot e\binom{bc}{ba}+\frac{dc}{bc}\cdot f\binom{bc}{dc}=1. \tag{q_2}$$

Eliminating ab between (q_1), (q_2), the coefficient of dc is

$$f\binom{ad}{ab}\cdot f\binom{bc}{dc} - e\binom{bc}{ba}\cdot e\binom{ad}{cd},$$

which is zero.

For if the conic is

$$ad\cdot bc = m\cdot ab\cdot cd,$$

$$f\binom{ad}{ab}\cdot f\binom{bc}{dc} = m = e\binom{bc}{ba}\cdot e\binom{ad}{cd}.$$

Thus the eliminant is

$$e\binom{ba}{cb}\cdot bc = f\binom{ba}{ad}\cdot ad,$$

which, by (p), is the Pascal of

$$ebcfad, \tag{2}$$

the middle line of (g), which shows concurrency for (g).

By means of equations like (q_1) concurrency for (h) may be shown.

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[1] See Veronese, loc. cit.